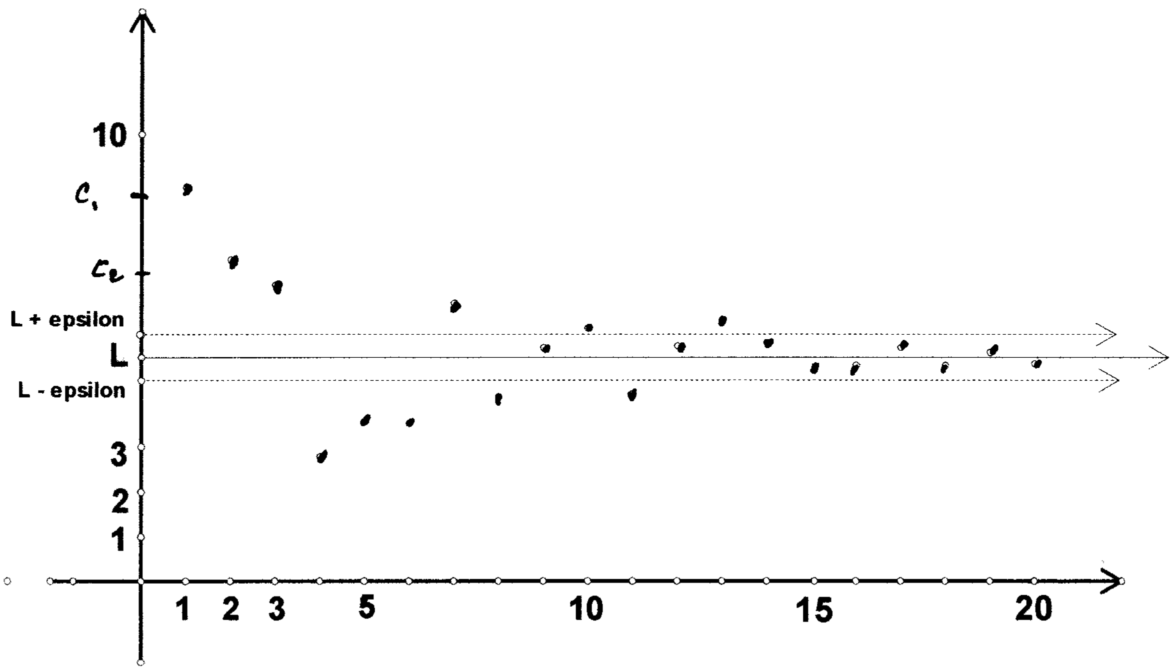
The Limit of a Sequence : Definitions

TAIL: Given a sequence $c_1,\ c_2,\ c_3,\ \ldots,\ c_K,\ c_{K+1},\ \ldots$

a **TAIL** of the sequence is formed by a term c_K and all the terms after it.

CONVERGE: A sequence $\{ c_n \}$ **converges to the Limit L** if, for every tiny measure of closeness, ε, there is some tail of the sequence that stays that close to the number L. In this case we say "The limit of c_n as n approaches infinity is L" or "As n approaches infinity, c_n approaches L."

We also write: $\lim_{n \to \infty} c_n = L$ or $c_n \to L \ \ as \ \ n \to \infty$.



DIVERGE: If a sequence fails to converge to a limit, it is said to **diverge**.

The sequences $\{ a_n = 1/n \}$ and $\{ b_n = (n+1)/n \}$ both converge to limits:

$$\lim_{n \to \infty} \frac{1}{n} = L = 0 ; \qquad \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1 + 0 = 1$$

The sequences $\left\{ c_n = n^2 + 3n \right\}$ and $\left\{ d_n = (-1)^n \right\}$ both diverge.

We say $c_n \to \infty \ \ as \ \ n \to \infty$ if, for every $N > 0$, a tail stays greater than N.

We say $c_n \to -\infty \ \ as \ \ n \to \infty$ if, for every $N > 0$, a tail stays more negative than $(-N)$.

In both cases, the sequence $\{ c_n \}$ diverges.

Theorems: If $a_n \to L$ and $b_n \to M$ then :

1) $a_n b_n \to L\,M$; 2) $a_n \pm b_n \to L \pm M$; 3) $\sqrt{a_n} \to \sqrt{L}$ *if* $L \geq 0$.

4) $\dfrac{a_n}{b_n} \to \dfrac{L}{M}$ *if* $M \neq 0$; 5) $f(a_n) \to f(L)$ **if f is continuous at $x = L$.**

Example: $\cos(a_n) \to \cos(L)$ **and** $\ln(b_n) \to \ln(M)$ *if* $M > 0$.

6) **If $\{ a_n \}$ is bounded above and below and $b_n \to \infty$, then** $\frac{a_n}{b_n} \to 0$

Example: $\dfrac{(-1)^n\, 10^8}{n\sqrt{n+2}} \to 0$

7) **If $a_n \to -\infty$, then** $e^{(a_n)} \to 0$, **but if $a_n \to 0$, then** $e^{(a_n)} \to e^0 = 1$.

8) **If $\{ a_n > 0 \}$ and a_n do not converge to 0 and $b_n \to 0$, then** $\frac{a_n}{b_n} \to \infty$

9) **If $\{ a_n < 0 \}$ and a_n do not converge to 0 and $b_n \to 0$, then** $\frac{a_n}{b_n} \to -\infty$

10) **If $a_n \to \pm\infty$ and $b_n \to \pm\infty$, then we must convert** $\frac{a_n}{b_n}$ **to a different form.**

11) **If $a_n \to 0$ and $b_n \to 0$, then we must convert** $\frac{a_n}{b_n}$ **to a different form.**

12) **An alternating sequence converges to L = 0, <u>if it converges at all</u>.**

Examples:

$$\frac{\sqrt{n+3}}{\sqrt{4n+5}} = \frac{\sqrt{n}\sqrt{1+(3/n)}}{\sqrt{n}\sqrt{4+(5/n)}} = \frac{\sqrt{1+(3/n)}}{\sqrt{4+(5/n)}} \to \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$\frac{3n^2+5n}{4 - 2n^2} = \frac{n^2\,(3+(5/n))}{n^2\,((4/n^2)-2)} = \frac{(3+(5/n))}{((4/n^2)-2)} \to \frac{3}{-2} = -\frac{3}{2}$$